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DEFLECTIONS OF POSTBUCKLED UNLOADED ELASTO–PLASTIC THIN VERTICAL COLUMNS

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Abstract—Thin vertical columns form a curve with large deflections and slopes when loaded axially by larger than the critical buckling load. For thin columns of elasto–plastic properties, these loads could induce either an elastic state throughout the length of the bar, or an elasto–plastic state with a moment at the base of the bar larger than the yield moment but smaller than the plastic moment. As the load increases, deflections and slopes of the median line of the column increase and yielding progresses from the base towards the free end. However, it is possible that the moments along the median line of the column may decrease, i.e. the column unloads. Due to the large curvatures of the column, the exact expression for the curvature of the median line is used in the Bernoulli–Euler equation (Theory of the Elastica). The boundary-value problem that results contains nonlinear ordinary differential equations. The Euler–Gauss predictor–corrector method has been used to solve the nonlinear differential equations numerically. Deflections of unloaded elasto–plastic thin columns are presented in a graphical form and compared with those of a linearly elastic material.

NOTATION

The folowing symbols are used in this paper:

- E Young's modulus of elasticity
- h height of rectangular cross-section
- I moment of inertia of rectangular cross-section about the weak axis
- j pivotal point on the elastic curve of the column
- L length of the column along the median line
- M bending moment
- M_f bending moment at the base of the column
- M_j bending moment at pivotal point j
- M_n bending moment at pivotal point j after unloading
- M_o bending moment at pivotal point j preceding unloading
- M_p plastic moment
- M_y yield moment
- P point load acting at the free end of the column
- P_{cr} critical buckling load, Euler's buckling load
- P_m point load just before instability occurs
- s arc length along the median line of the column measured from the origin
- x, y vertical and horizontal coordinates, respectively
- x_b, y_b x_{max}, y_{max}, vertical and horizontal coordinates of the free end of the column, respectively.
- U_e unloading of the elastic section
- U_{ey} unloading of the elastic and yielded sections
- α angle between tangent to the elastic curve and the vertical at the free end of the column
- ϵ strain in/in
- ϵ_y strain at yield in/in
- θ slope of the elastic curve at s
- θ_y slope at the end of the yielded portion of the column
- $\phi = \dot{\theta}$ curvature $= d\theta/ds$
- $\phi_y = \dot{\theta}_y$ curvature at yield
- $\phi_j = \dot{\theta}_j$ $(d\theta/ds)$ at pivotal point j
- ϕ_n curvature at pivotal point j after unloading
- ϕ_o curvature at pivotal point j preceding unloading
- σ stress
- σ_y yield stress
- λ constant $= \dfrac{E}{\sigma_y}\dfrac{h}{L}$
- Δs step size

INTRODUCTION

Problems involving large finite deflections of thin bars, i.e. the elastica, have been treated by numerous investigators over a period extending up to the present time. The elastica is the

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deflected shape of the centerline of a thin bar found by using the exact expression of the curvature in the Bernoulli–Euler moment–curvature relation, M–ϕ. Several applications of the elastica theory can be found in the monograph by Frisch-Fay[1]. These applications are limited to bars of linear elastic material. Also, Eisley[2] has given a comprehensive list of references on the subject.

Oden and Childs[3] examined the problem of finite deflections of a clamped-end thin column constructed of a nonlinearly elastic material represented by a moment-curvature relation in the form of a hyperbolic tangent law.

Prathap and Varadan[4] have studied the inelastic large deformation of a uniform cantilever of rectangular cross section with a vertical point load at the free end where the material of the beam is assumed to have stress–strain laws of the Ramberg–Osgood type.

Lo and Gupta[5] examined the bending of a nonlinear rectangular beam in large deflections where the linear stress–strain relationship is used for sections of the beam that deformed elastically, and by a logarithmic function of strain for sections where the maximum stress exceeds the elastic limit.

Axially loaded thin columns, originally straight, could buckle into a plane curve with large curvatures when the load, P, is larger than the critical buckling load, P_{cr}. Figure 1 shows the deformed configurations for axially loaded thin elastic column for various values of α and P/P_{cr} (the elastica)[6].

Thin columns of elastic–plastic material loaded axially with loads larger than P_{cr} as shown in Fig. 2 could exhibit either an elastic state throughout the length of the bar, or an elasto–plastic state with a moment at the base of the bar larger than the yield moment but smaller than the plastic moment. As the load increases, deflections and slopes of the median line of a thin column increase and yielding progresses from the base of the bar towards the other end. However, it is possible that the moments along the median line of a largly deflected column may be reduced in some sections, i.e. a region of the column is "unloading". A complete analysis of the unloading phenomenon is given in the section of "Discussion of Results".

This paper presents the unloading behavior of postbuckled vertical thin columns of an elasto–plastic material. Deflections are presented in a graphical form and compared with those of linear elastic thin columns (the elastica).

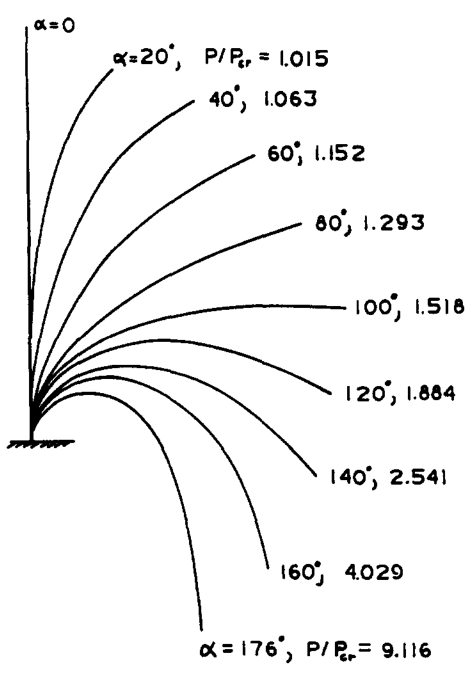

Fig. 1. Deflected forms of the elastica for various values of α and P/P_{cr}. This figure is from Ref. [6], Timoshenko and Gere, Theory of Elastic Stability, p. 80 (1961).

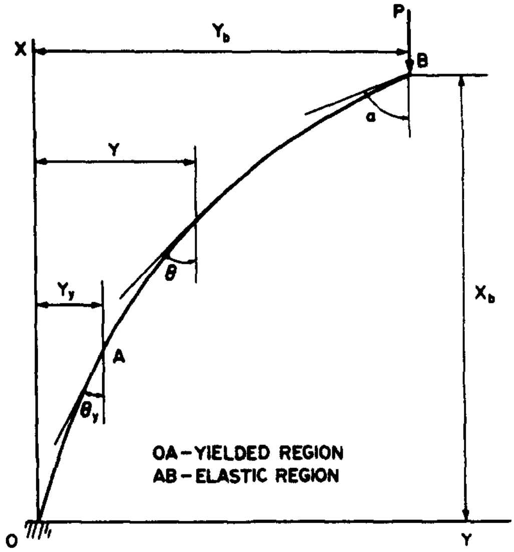

Fig. 2. Postbuckled elasto-plastic thin column.

PROBLEM STATEMENT

A thin column of rectangular cross-section fixed at the base and axially loaded at the free end as shown in Fig. 2 is considered. The material of the column is assumed to follow an elasto-plastic stress-strain relationship as shown in Fig. 3, where:

$$\sigma = E\epsilon \qquad 0 < \epsilon < \epsilon_y \tag{1}$$

and

$$\sigma = \sigma_y \qquad \epsilon_y < \epsilon < \infty \tag{2}$$

in which σ is the stress, ϵ is the strain and E is the Young's modulus.

In this problem, it is assumed that the weight of the column is negligible; axial and shearing stresses are small compared to those induced by bending; strains stay small; and strain-hardening is negligible.

MOMENT-CURVATURE RELATIONS

Due to the deformed configuration of a thin column under loads larger than the critical buckling load as shown in Fig. 1, geometric nonlinearity arises since the bar has large

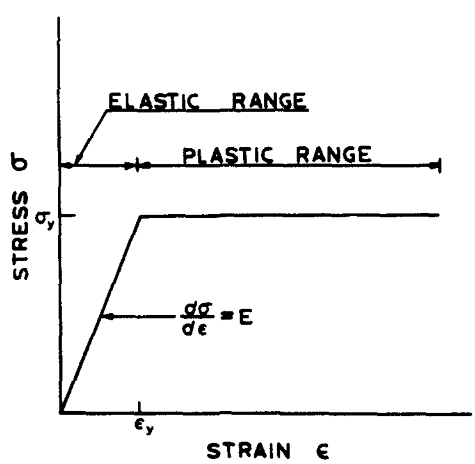

Fig. 3. Stress-strain relationship for a thin column.

deflections and slopes, accordingly, problems involving deflection of bars of significant curvature cannot be treated within the framework of the linear bending theory; therefore, the exact expression for the curvature of the median line of the column must be used in the Bernoulli–Euler equation.

Three moment–curvature relationships are required to develop the analysis of postbuckled unloaded elasto–plastic thin columns. These relations are:

1. For the linear elastic section of the column the equation

$$\phi = \frac{d\theta}{ds} = \frac{M}{EI} \qquad \phi < \phi_y \tag{3}$$

is applicable when the deflected median line of the column is given in the intrinsic form $s = f(\theta)$, where s is the coordinate measured along this line and θ is the slope at s. EI is the instantaneous flexural rigidity with respect to the weak axis and M is the bending moment.

2. For an elasto–plastic state of stress of a rectangular cross-section in which no unloading occurs, the M–ϕ relationship is given in [7 and 8]. This relationship can be written as follows:

$$\phi = \frac{d\theta}{ds} = \phi_y M_y^{1/2} \cdot (3M_y - 2M)^{-1/2} \quad \phi > \phi_y \tag{4}$$

in which M_y is the yield moment and ϕ_y is the curvature at the onset of yielding. The nondimensional M–ϕ curve is shown in Fig. 4.

3. For an elasto–plastic thin column where unloading takes place, the elastic section unloads according to eqn (3), while the yielded portion unloads linearly according to the following equation.

$$\phi_n = M_n/EI + \phi_o - M_o/EI \quad \phi > \phi_y \tag{5}$$

in which ϕ_n, M_n are the curvature and moment, respectively, along the median line of the yielded section after unloading and ϕ_o, M_o are the curvature and moment preceding unloading. A family of moment–curvature curves representing eqn (5) is shown in Fig. 5.

In general the solution of problems involving large deflections of bars is one of considerable methematical difficulty and closed-form solutions are available only in few cases [1, 6]. Due to the complexity of the problem, a step-by-step forward numerical intergration procedure, the Euler–Gauss predictor–corrector method [9–11] has been used to solve the nonlinear boundary-value problem.

NUMERICAL METHOD

Equations (3)–(5) can be written in the following manner:

$$\dot{\theta}_j = M_j/EI \qquad \dot{\theta}_j < \dot{\theta}_y \tag{6}$$

$$\dot{\theta}_j = \dot{\theta}_y M_y^{1/2}(3M_y - 2M_j)^{-1/2} \qquad \dot{\theta}_j > \dot{\theta}_y \tag{7}$$

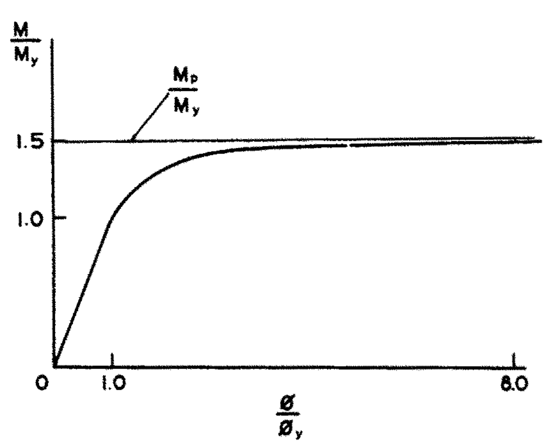

Fig. 4. Nondimensional moment–curvature relationship for a rectangular cross-section.

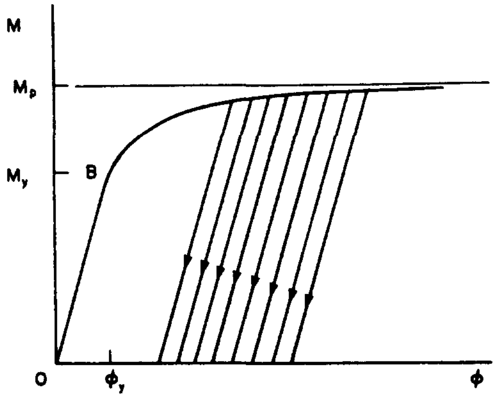

Fig. 5. Moment-curvature relationship for a rectangular cross-section for loading and unloading.

and

$$\dot{\theta}_j = M_j/_{EI} + \dot{\theta}_{j-1} - M_{j-1}/_{EI} \qquad \dot{\theta}_j > \dot{\theta}_y \tag{8}$$

in which j = pivotal point on the median line of the column, $\dot{\theta}_j$ = curvature = $d\theta/ds$ and M_j = bending moment at j.

Equation (8) shows that it is essential for an elasto-plastic state of stress at the onset of unloading in the yielded section, one must consider the preceding state of stress just before unloading, in order to compute the exact values of the moments and curvatures.

To initiate the solution of the nonlinear, first order differential eqns (6)–(8) by using the Euler–Gauss method, one must consider the initial boundary values at the base of the column, i.e.

$$\theta(0) = 0 \tag{9}$$

and

$$y(0) = 0. \tag{10}$$

To determine the deflections of the column along the entire length of its median line for a load $P > P_{cr}$, an initial trial value for the deflection at the free end of the column is assumed. This will fix the moment at the base of the column. Using eqns (6)–(10) a step-by-step forward intergration method is used to move along the median line from the base to the free end of the column in small increments Δs finding, at each step, new values for the curvature, slopes and deflections. At the end of each cycle the computed deflection at the free end is compared with the initial trial value. If the computed value is close enough to the initial value, within an acceptable tolerance, a solution is obtained. Otherwise, the arithmetical average of the computed and the trial value is found and is used as a new trial value for the deflection at the free end. The process is repeated until a solution is obtained. A knowledge of the nature of the elastica problem is most helpful in choosing the initial trial value for the deflection at the free end of the column. The convergence is noted to be relatively slow for all loads.

The numerical algorithm outlined above was performed on the UNIVAC 1110.

DISCUSSION OF RESULTS

Figure 1 shows several deformed configurations of a thin column for various values of $P > P_{cr}$. These configurations are in a stable state of equilibrium when the stresses throughout the entire length of the bar remain below the elastic limit[6]. However, a quite different behavior is encountered for thin columns of an elasto-plastic state of stress. This behavior can be summarized as follows:

1. The bar remains stable for loads P larger than P_{cr} (about 1.152 P_{cr}) and becomes unstable when the moment at the base of the bar approaches the plastic moment of the cross-sectional area. This instability occurs as a result of the reduced stiffness of the material after yielding. In this case, the end slopes α are usually small (less than 60°) and no unloading occurs.

2. The column remains stable for loads P larger than P_{cr}, however, as the load P reaches P_m

(the load which causes instability) and α is less than 90°, the column undergoes a large deflection and becomes stable at a position with $\alpha > 90°$. Instability occurs again when the moment at the base M_f approaches M_p. The column does not exhibit unloading and its behavior was discussed in Ref. [12].

3. For columns that remain stable for loads $P > P_{cr}$, it is possible that the moments in some regions decrease, i.e. the column unloads, as load P and deflection increase. This behavior occurs only when $\alpha > 90°$. Only the behavior of these columns is examined in this paper. Columns may unload in two ways:

(a) The column may unload only in its elastic section according to eqn (3) while the yielded section (close to the base) may keep loading until the moment at the base approaches the plastic moment when the column becomes unstable; or

(b) The column may unload both in its elastic and yielded sections. In this case the elastic section unloads according to eqn (3) while the yielded section unloads according to eqn (5). As load increases, unloading continues in the elastic and yielded sections, however, sections close to the base may still load until instability occurs when M_f approaches M_p.

Table 1 shows values of the bending moments along the median line of an elasto-plastic unloaded thin column for several values of P/P_{cr}. It is shown that the bending moments in the unloaded region of the column, those below the dashed lines, decrease as load increases. When unloading occurs, the column unloads from its tip to some pivotal point along the median line, such as $j = 381$, 221, 151, 131 and 121. As load increases and unloading takes place in the

Table 1. Moments in N-M along the median line of unloaded thin column for various values of P/P_{cr}

L = 1270 mm (50 in)			M_y = 0.276 N-M (2.441 in-Lb)			
L/h = 1600			P_{cr} = 0.1675 N (0.03765 Lb)			
λ = 0.625						
P/P_{cr}	1.884	2.00	2.30	2.35	2.40	2.45
Type of Unloading	-	U_e*	U_e	U_{ey}	U_{ey}	U_{ey}
Unloading from j to tip of column	-	381	221	151	131	121
j	Moment					
1	0.322(2.853)	0.340(3.007)	0.375(3.323)	0.380(3.365)	0.384(3.401)	0.388(3.438)
51	0.315(2.786)	0.331(2.929)	0.362(3.201)	0.365(3.223)	0.368(3.259)	0.371(3.284)
101	0.293(2.596)	0.306(2.710)	0.327(2.890)	0.328(2.904)	0.329(2.912)	0.330(2.918)
121	0.282(2.493)	0.293(2.594)	0.309(2.736)	0.310(2.743)	0.310(2.745)	0.310(2.743)
131	0.275(2.437)	0.286(2.531)	0.300(2.654)	0.300(2.659)	0.300(2.658)	0.300(2.653)
151	0.262(2.317)	0.271(2.398)	0.281(2.486)	0.281(2.485)	0.280(2.479)	0.279(2.470)
201	0.224(1.986)	0.230(2.036)	0.232(2.055)	0.231(2.045)	0.229(2.030)	0.227(2.013)
221	0.209(1.846)	0.213(1.886)	0.213(1.885)	0.212(1.872)	0.210(1.855)	0.207(1.836)
251	0.185(1.634)	0.188(1.661)	0.185(1.636)	0.183(1.621)	0.181(1.603)	0.179(1.584)
301	0.145(1.282)	0.146(1.293)	0.141(1.249)	0.139(1.233)	0.137(1.215)	0.135(1.198)
351	0.107(0.943)	0.107(0.945)	0.101(0.898)	0.100(0.885)	0.098(0.868)	0.097(0.855)
381	0.084(0.746)	0.084(0.745)	0.079(0.703)	0.078(0.693)	0.077(0.680)	0.075(0.664)
401	0.070(0.618)	0.070(0.617)	0.065(0.579)	0.064(0.571)	0.063(0.556)	0.062(0.550)
451	0.034(0.305)	0.034(0.304)	0.032(0.284)	0.032(0.281)	0.031(0.270)	0.030(0.264)
501	0.00	0.00	0.00	0.00	0.00	0.00

* U_e = unloading of the elastic section

U_{ey} = unloading of the elastic and yielded sections

Table 2. Ratios of maximum deflections/length and M_f/M_y of unloaded elasto-plastic thin columns for various values of P/P_{cr}

L = 1270 mm (50 in.), L/h = 800 $\lambda = 0.750$

									Type of Unloading			
									U_e (391)	U_e (191)	U_{ey} (121)	U_{y} (91)*
P/P_{cr}	1.015	1.063	1.10	1.152	1.293	1.400	1.518	1.600	1.750	1.884	1.930	2.000
M_f/M_y	0.20	0.41	0.51	0.63	0.86	0.93	1.11	1.19	1.33	1.41	1.43	1.46
Ymax/L	0.220	0.422	0.505	0.593	0.719	0.765	0.793	0.807	0.819	0.810	0.803	0.789
Xmax/L	0.970	0.881	0.823	0.741	0.560	0.451	0.343	0.262	0.094	-0.068	-0.120	-0.202
α	20°	40°	49°	60°	80°	90°	100°	107°	120°	131°	134°	139°

L = 1270 mm (50 in.), L/h = 1600 $\lambda = 0.625$

									Type of Unloading				
									U_e (381)	U_e (221)	U_{ey} (151)	U_{ey} (131)	U_{ey} (121)*
P/P_{cr}	1.015	1.063	1.100	1.152	1.293	1.400	1.518	1.750	1.884	2.000	2.300	2.350	2.400	2.450
M_f/M_y	0.17	0.34	0.43	0.53	0.72	0.82	0.93	1.09	1.17	1.23	1.36	1.38	1.39	1.41
Ymax/L	0.220	0.422	0.505	0.593	0.719	0.765	0.792	0.806	0.804	0.799	0.767	0.760	0.754	0.744
Xmax/L	0.970	0.881	0.823	0.741	0.560	0.451	0.349	0.193	0.109	0.037	-.143	-0.173	-0.199	-0.230
α	20°	40°	49°	60°	80°	90°	100°	114°	121°	126°	139°	141°	143°	145°

* Numbers enclosed in parentheses indicate pivotal points along the median line of the bar.

column, the elastic section of the column unloads first, then unloading of the yielded section proceeds as shown for P/P_{cr} of 2.35, 2.40 and 2.45.

Table 2 shows values of M_f/M_y, Y_{max}/L and X_{max}/L for various values of P/P_{cr} of an elasto–plastic thin bar with $\lambda = (E/\sigma_y)(h/L) = 0.625$ and 0.750. When stresses in the bar remain below the elastic limit, i.e. $M_f/M_y < 1.00$, then the y- and x-deflections are similar to those of the elastica. However, as M_f/M_y becomes >1.00, the deflections of the bar deviate from those of the elastica.

Figure 6 shows the x-deflection at the free end/length, X_{max}/L, vs P/P_{cr} for several values of λ. The solid curve represents the classical elastica, while the dashed curves represent elasto–plastic thin columns. The curves coincide with the elastica where all stresses are below the elastic limit and starts deviating when the base yields. Stable columns with values of λ between 0.50 and 1.00 exhibit unloading phenomenon and their x-deflections deviate largely from the

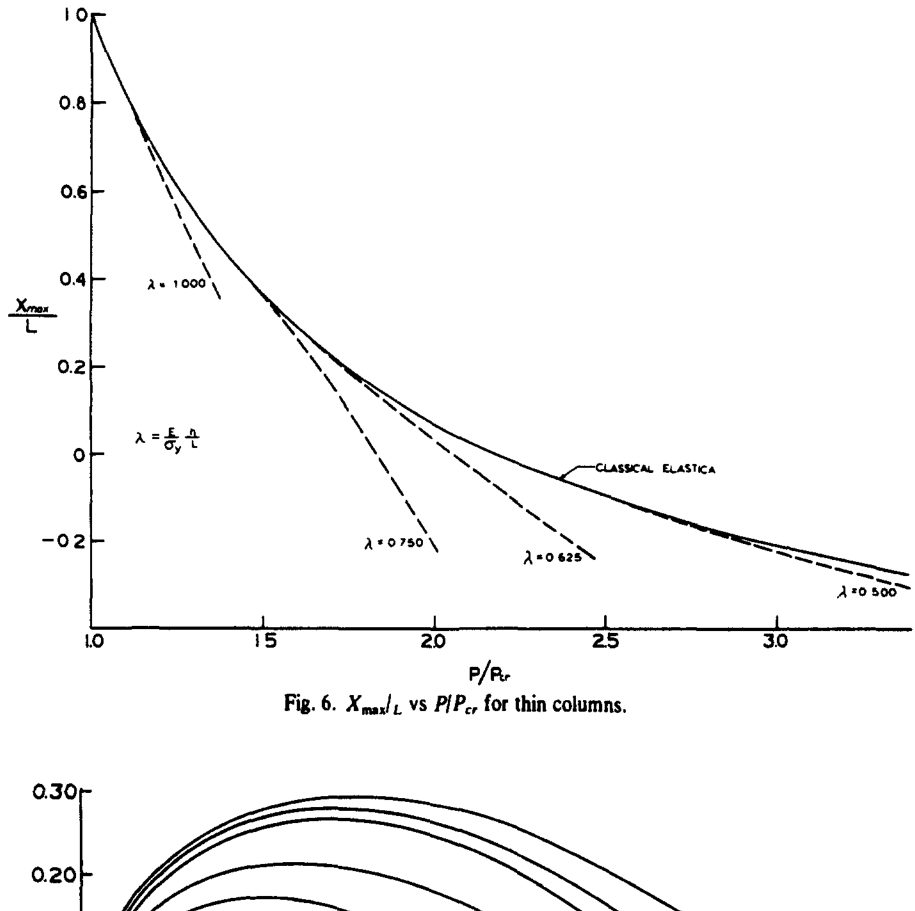

Fig. 6. X_{max}/L vs P/P_{cr} for thin columns.

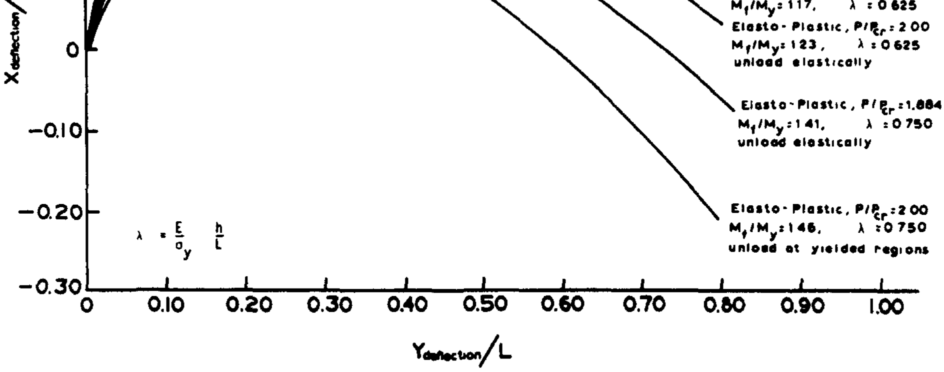

Fig. 7. Deflected forms of thin columns for various values of λ.

classical elastica. For values of $\lambda > 1$, the columns behave as indicated in item 1, while columns with values of $\lambda < 0.50$ behave closely to the classical elastica. The dashed curves are terminated when the elasto-plastic columns become unstable as M_f approaches M_p.

The y-deflection at the free end for elasto-plastic thin columns is found to be very close to the one of the elastica, as indicated in Fig. 7 where several deformed configurations of thin columns are shown. It is noticed that for the same values of P/P_{cr}, deflections and slopes increase as λ and M_f/M_y increase.

SUMMARY AND CONCLUSIONS

The unloading behavior of elasto-plastic postbuckled thin columns is examined in this paper and their deflections are compared to those of the classical elastica. The following conclusions are drawn from this study:

1. Columns with values of λ between 0.50 and 1.00 exhibit unloading phenomenon and their x-deflections deviate largely from those of the elastica. These columns remain stable for loads larger than P_{cr}. However, they become unstable when the moment at the base of the column approaches M_p.

2. Columns with values of $\lambda > 1$ do not exhibit unloading and they become unstable when M_f approaches M_p.

3. For $\lambda < 0.50$ the columns behave closely to the elastica and they remain always in a stable state equilibrium.

4. The dashed curves are terminated when the column becomes unstable as the moment M_f approaches M_p.

5. The y-deflection at the free end for elasto-plastic column is found to be very close to the one of the elastica.

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